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Transformations of mechanical systems with cyclic coordinates and new integrable problems

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Abstract
We show that the well-known Routhian procedure of ignoring cyclic coordinates is far more than a tool for obtaining equations of motion of reduced order in a Lagrangian form. A transformation is introduced that involves a number of arbitrary functions or additional parameters in the system while preserving the Routhian equations of motion. Although these parameters invoke new physical effects in the transformed system, the solution of the latter is always obtained in a simple way from that of the original system. In particular, from any integrable system with k cyclic degrees of freedom we obtain a family of systems integrable on a fixed level of the cyclic integrals and physically generalizing that system through the inclusion of k arbitrary functions that depend only on the noncyclic coordinates. In many problems of physical interest, a general integrable case can also be generalized to an integrable case for arbitrary initial conditions. The method is applied to some problems of rigid body dynamics. Four new integrable problems are obtained as generalizations of known cases by including certain additional combinations of gravitational and electromagnetic forces. The new cases are presented in an explicit way that enables direct verification of the constancy of the integrals using the equations of motion. Explicit time solution of the new cases is discussed. Physical interpretation is given for two cases.

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1. Introduction

Integrable problems are rare exceptions in the totality of problems in mechanics. For these one can make many important assertions about the global behaviour of motion and in many cases the general explicit solution of the equations of motion in terms of time can be obtained. Integrable problems are also of great importance in the study of real physical non-integrable systems near to them. The search for integrable mechanical systems will remain one of the principal fields of investigation in mechanics, astronomy, physics, engineering and other sciences.

In dealing with a concrete dynamical problem, it is usually hard to decide whether it is integrable or not. Even if there is some numerical evidence of the regularity of motion, this does not mean necessarily that the system is integrable unless the complementary integrals can be found explicitly. Thus, there appeared a great interest in proving that a certain given system is non-integrable and clarifying the obstructions to integrability. This direction was founded by Poincare and has undergone remarkable developments recently. It was particularly effective in a limited number of problems of rigid body dynamics (see e.g. [1–6]) and of celestial mechanics. However, research in this direction has not succeeded in isolating new integrable cases as much as it has demonstrated the rarity of integrable cases in that field.

A large collection of integrable systems in the plane is presented in the review [7]. The three classical integrable cases of the simple heavy rigid body dynamics known after Euler, Lagrange and Kovalevskaya and their generalizations to the case of a gyrostat [8] are examples of a unique type of integrable system, whose configuration space is not flat. The same applies for all 16 known integrable cases of the problem of motion of a rigid body and their subsequent generalizations [9] (see [10] for the 16th case). All these systems share the property that the complementary integrals are polynomials in the velocity variables. However, they were obtained by diverse methods. Explicit solutions were obtained until now only for a very few and restricted cases of those systems (see e.g. [11]).

The inverse method developed in [12, 13] has proved very successful in constructing a large number of new integrable mechanical systems in a unified systematic way. Some of these systems generalize previously known integrable problems and many others are completely new. For example, several-parameter integrable systems are obtained, which are reduced to known integrable cases of rigid body dynamics for some choice of the parameters and to integrable systems on the ellipsoid, sphere, or in the plane for some other choices. No physical interpretation was found for most of those new integrable cases. Their dynamical behaviour and their explicit solutions in terms of time are also unknown.

In [9] we have introduced another inverse method, specially designed for rigid body dynamics. Certain transformations were found that leave invariant the form of the Euler–Poisson equations of motion. Five general and 15 conditional integrable cases were obtained using these transformations (see also [14]).

In this paper we present another inverse method for generalizing known integrable cases of a certain type of mechanical system, namely those systems whose structure involves cyclic coordinates. The basic idea is that for such a system to be integrable, all that matters is the structure of its Routhian equations of motion after ignoring the cyclic coordinates. We use a simple observation that several Lagrangian mechanical systems that have different Lagrangian and Routhian functions can be reduced to one and the same set of Routhian equations. Clearly, this will be the case if the Routhians of these systems differ only by constant terms that may depend only on the cyclic constants.

This situation is utilized to introduce certain transformations of the systems under consideration, which preserve their Routhian equations of motion. This means, in particular, that the integrable cases of these systems are transformed into integrable cases of other more general systems involving a number of arbitrary functions or, at least, some additional parameters. It turned out that this generalization is nontrivial from the physical point of view and can be interpreted through the introduction of new forces of gyroscopic and electromagnetic origin in the system.

This method is applied here in detail to problems involving a rigid body or a gyrostat acted upon by asymmetric and skew fields, to which the previous method was not applicable. Several new integrable cases of those problems are found.

2. Transformations of cyclic velocities

Consider the mechanical system of $n + k$ degrees of freedom, of which k degrees are cyclic, characterized by the time-independent Lagrangian

$$L = L(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n, \dot{Q}_1, \ldots, \dot{Q}_k). \tag{1}$$

The system admits the cyclic integrals

$$\frac{\partial L}{\partial \dot{Q}_i} = f_i \qquad i = 1, \ldots, k. \tag{2}$$

Let us consider another system with the Lagrangian

$$L' = L(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n, \dot{Q}'_1 + \nu_1, \ldots, \dot{Q}'_k + \nu_k) - \sum_{i=1}^{k} \beta_i \nu_i (q_1, \ldots, q_n) \tag{3}$$

where β_i are certain constants and ν_i are certain functions of the palpable coordinates $q_1, \ldots, q_n$. We note that the system (3) is time independent with the cyclic variables $Q'_1, \ldots, Q'_k$.

This system can be considered as a transformation of (1) through the linear time-independent transformation of the cyclic variable rates

$$\dot{Q}_i = \dot{Q}'_i + \nu_i (q_1, \ldots, q_n). \tag{4}$$

Consider the motion of the system (3) on the same level of cyclic integrals as in (2), i.e.

$$\frac{\partial L'}{\partial \dot{Q}'_i} = f_i \qquad i = 1, \ldots, k. \tag{5}$$

This is the transformed form of (2) according to (4).

Now, let R and R' be the Routhians of the two systems, then their difference

$$\begin{aligned} R' - R &= L - \sum_{i=1}^{k} \beta_i \nu_i - \sum_{i=1}^{k} \dot{Q}'_i f_i - \left(L - \sum_{i=1}^{k} \dot{Q}_i \; f_i \right) \\ &= \sum_{i=1}^{k} (\dot{Q}_i - \dot{Q}'_i) f_i - \beta_i \nu_i \\ &= \sum_{i=1}^{k} (f_i - \beta_i) \nu_i. \end{aligned} \tag{6}$$

The Routhian equations of motion (see e.g. [15] and [16]) of the system characterized by (1) and (2) will be identical to those obtained for the transformed system (3) and (5) if we set $\{f_i = \beta_i, i = 1, \ldots, k\}$. In other words, under the last conditions, the arbitrary functions ν_i do not affect the solution for the non-cyclic coordinates.

From the above considerations we draw the following theorems:

Theorem 1. *If the system with the Lagrangian*

$$L = L(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n, \dot{Q}_1, \ldots, \dot{Q}_k) \tag{7}$$

is integrable for arbitrary initial conditions, then the system whose Lagrangian is

$$L' = L(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n, \dot{Q}'_1 + \nu_1, \ldots, \dot{Q}'_k + \nu_k) - \sum_{i=1}^{k} \beta_i \nu_i (q_1, \ldots, q_n) \tag{8}$$

is integrable for arbitrary functions ν_i and arbitrary constants $\{\beta_i\}$ on the level

$$\left\{\frac{\partial L'}{\partial \dot{Q}'_i} = \beta_i \qquad i = 1, \ldots, k\right\} \tag{9}$$

of the cyclic integrals.

Theorem 2. *If $\{q_1(t), \ldots, q_n(t), Q_1(t), \ldots, Q_k(t)\}$ is any solution of the system described by (7), then $\{q_1(t), \ldots, q_n(t), Q_1(t) - \int \nu_1(q_1(t), \ldots, q_n(t))\, \mathrm{d}t, \ldots, Q_k(t) - \int \nu_k(q(t), \ldots, q_n(t))\, \mathrm{d}t\}$ is a solution of the system with Lagrangian (8) for arbitrary functions ν_i.*

Theorem 3. *The arbitrary functions $\{\nu_i\}$ do not affect in any way the explicit solution of the Lagrangian equations of motion derived from (8) with respect to the palpable part of the variables, since those variables are determined by the Routhian which is independent of $\{\nu_i\}$.*

Throughout the present paper we will call a problem *general integrable* if it is integrable for arbitrary initial conditions and *conditional integrable* if it is integrable only on a single level $\{f\}$ of the cyclic integrals but for all initial conditions compatible with that level. In both types of problems the solution can be reduced to quadratures through the application of Liouville's theorem to the reduced n-dimensional Hamiltonian system. It is thus sufficient to point out n time-independent first integrals in involution to ensure integrability in these cases.

It should be stressed that the integrability of the system with Lagrangian (8) is conditional, i.e. valid only for initial conditions consistent with the restriction (9), even if the original system (7) is integrable for arbitrary initial conditions. There are, however, very important situations when the new system can be made integrable for all initial conditions. This depends only on the structure of the potential part of the Lagrangian. Such situations will be discussed for generalized natural systems in section 4.

3. The case of a generalized natural system

For the sake of clarity and for future applications we consider in detail the case of a generalized natural system with three degrees of freedom, of which one is cyclic. Let

$$L = \tfrac{1}{2}\left(a_{11}\dot{q}_1^2 + 2a_{12}\dot{q}_1\dot{q}_2 + a_{22}\dot{q}_2^2\right) + (c_1\dot{q}_1 + c_2\dot{q}_2)\dot{Q} + \tfrac{1}{2}c_3\dot{Q}^2 + b_1\dot{q}_1 + b_2\dot{q}_2 + b_3\dot{Q} - V \tag{10}$$

where a_{ij}, b_i, c_i, V depend only on q_1, q_2, so that Q is a cyclic variable. On an arbitrary level of the cyclic integral

$$c_1\dot{q}_1 + c_2\dot{q}_2 + c_3\dot{Q} + b_3 = f \tag{11}$$

the Routhian has the form

$$R = \frac{1}{2}\left(a_{11}\dot{q}_1^2 + 2a_{12}\dot{q}_1\dot{q}_2 + a_{22}\dot{q}_2^2\right) - \frac{1}{2c_3}[c_1\dot{q}_1 + c_2\dot{q}_2 + b_3 - f]^2 + b_1\dot{q}_1 + b_2\dot{q}_2 - V. \tag{12}$$

Now we perform in (10) the transformation

$$\begin{aligned} L' &= L - f\nu \\ \dot{Q} &= \nu + \dot{Q}' \qquad \nu = \nu(q_1, q_2). \end{aligned} \tag{13}$$

According to the previous section, we get the new Lagrangian

$$L' = \tfrac{1}{2}\left(a_{11}\dot{q}_1^2 + 2a_{12}\dot{q}_1\dot{q}_2 + a_{22}\dot{q}_2^2\right) + (c_1\dot{q}_1 + c_2\dot{q}_2)\dot{Q}' + \tfrac{1}{2}c_3\dot{Q}'^2 + b'_1\dot{q}_1 + b'_2\dot{q}_2 + b'_3\dot{Q}' - V' \tag{14}$$

where

$$\begin{aligned} b_i' &= b_i + \nu c_i \qquad i = 1, 2, 3 \\ V' &= V + (f - b_3)\nu - \tfrac{1}{2}\nu^2 c_3. \end{aligned} \tag{15}$$

Thus, the transformation (13) has led only to a change in the potential and in the coefficients of the linear terms of the Lagrangian. This affects only the terms that express potential and gyroscopic forces in the Lagrangian equations of motion. In many concrete cases we find it easy to interpret the additional terms by means of the introduction of classical interactions: Newtonian, Coulomb, magnetic and Lorentz.

Now, we ignore the cyclic variable Q' in (14) with the aid of the cyclic integral

$$c_1\dot{q}_1 + c_2\dot{q}_2 + c_3\dot{Q}' + b_3' = f \tag{16}$$

and thus obtain the Routhian

$$R' = R \tag{17}$$

where R is the Routhian (12) on the same level f of the cyclic integral (11). Thus, the original and the transformed systems have the same set of Routhian equations of motion while describing different physical problems.

In certain circumstances, the transformation (13) can be used to simplify the Lagrangian equations of motion. For example, when the coefficients in (10) satisfy the conditions

$$b_i = -\nu c_i + \frac{\partial \chi}{\partial q_i} \qquad i = 1, 2, 3 \tag{18}$$

where χ is an arbitrary function in q_1, q_2 and $Q(q_3)$, the linear terms in (14) form a total time derivative $\frac{\mathrm{d}\chi}{\mathrm{d}t}$ and do not contribute to the equations of motion. In the transformed system the gyroscopic forces have disappeared. We say that the gyroscopic forces in (13) are reducible if for some function ν we have

$$\begin{aligned} &b_3 + \nu c_3 = \text{const} \\ &\frac{\partial(b_1 + \nu c_1)}{\partial q_2} - \frac{\partial(b_2 + \nu c_2)}{\partial q_1} = 0. \end{aligned} \tag{19}$$

In that case one can transform the problem in order to get rid of the gyroscopic forces at the expense of modifying the potential function.

4. Generalization of the general integrable problems

Now we consider a case of special interest. Let the potential V in (10) have the form

$$V = V_0 + \sum a_i v_i \tag{20}$$

where $\{a_i\}$ are arbitrary constants and V_0, v_i are certain functions in q_1, q_2. Let the system (10) be integrable. This means that, besides the Jacobi and the cyclic integrals, this system admits an integral, which will depend on the set of constants $\{a_i\}$, say

$$I_3 = F(q_1, q_2, \dot{q}_1, \dot{q}_2, \dot{Q}, a_1, a_2, \ldots). \tag{21}$$

If we choose ν in transformation (13) in the form

$$\nu = \sum n_i v_i \tag{22}$$

we get from (15) and (20)

$$\begin{aligned} V' &= V_0 + \sum a_i v_i + (f - b_3)\nu - \tfrac{1}{2}\nu^2 c_3 \\ &= V_0 + \sum (a_i + f n_i) v_i - b_3\nu - \tfrac{1}{2}\nu^2 c_3. \end{aligned} \tag{23}$$

The equations of motion for the transformed problem will admit the integral

$$I_3' = F\left(q_1, q_2, \dot{q}_1, \dot{q}_2, \dot{Q}' + \sum n_i v_i, a_1, a_2, \ldots\right). \tag{24}$$

Since this integral is valid for the arbitrary parameters $a_1, a_2, \ldots$ we can introduce a set of new arbitrary parameters $A_1, A_2, \ldots$ by the relations

$$A_i = a_i + f n_i \qquad i = 1, 2, \ldots \tag{25}$$

so that the potential can be written as

$$V' = V_0 + \sum A_i v_i - b_3 \nu - \tfrac{1}{2}\nu^2 c_3 \tag{26}$$

and the integral (24) as

$$I_3' = F\left(q_1, q_2, \dot{q}_1, \dot{q}_2, \dot{Q}' + \sum n_i v_i, A_1 - f n_1, A_2 - f n_2, \ldots\right). \tag{27}$$

The potential (26) now depends on the arbitrary parameters A_i which can be viewed as the actual parameters of the system, while the integral (27) depends on the arbitrary parameters A_i and on the cyclic constant f which can be replaced by its expression to obtain the final form

$$\begin{aligned} I_3' = F\Big(&q_1, q_2, \dot{q}_1, \dot{q}_2, \dot{Q}' + \sum n_i v_i, A_1 - n_1(c_1\dot{q}_1 + c_2\dot{q}_2 + c_3\dot{Q}' + b_3'), \\ &A_2 - n_2(c_1\dot{q}_1 + c_2\dot{q}_2 + c_3\dot{Q}' + b_3'), \ldots\Big). \end{aligned} \tag{28}$$

This means that the system with potential V' is integrable for arbitrary initial conditions and we are dealing with a case of general integrability.

Fortunately, in most known general integrable problems the potential has the structure (20) and then they admit generalization as general integrable problems involving as many arbitrary parameters as the number of constants $\{a_i\}$ in V. This result will be applied to several problems in the next sections.

5. On the dynamics of a rigid body gyrostat

Consider a rigid body in motion about its fixed point O. Let $OXYZ$ and $Oxyz$ be two Cartesian coordinate systems, fixed in space and in the body respectively. Let also $\boldsymbol{\omega} = (p, q, r)$ be the angular velocity of the body and $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}$ be the unit vectors in the directions of the XYZ-axes, all being referred to the body system which we take as the system of principal axes of inertia.

These variables can be expressed in terms of Euler's angles: ψ the angle of precession about the Z-axis, θ the angle of nutation (between the z and Z axes) and the angle of proper rotation φ about the z-axis. They have the form

$$\begin{aligned} \boldsymbol{\alpha} &= (\cos\psi\cos\varphi - \cos\theta\sin\psi\sin\varphi, -\cos\psi\sin\varphi - \cos\theta\sin\psi\cos\varphi, \sin\theta\sin\psi) \\ \boldsymbol{\beta} &= (\sin\psi\cos\varphi + \cos\theta\cos\psi\sin\varphi, -\sin\psi\sin\varphi + \cos\theta\cos\psi\cos\varphi, -\sin\theta\cos\psi) \\ \boldsymbol{\gamma} &= (\sin\theta\sin\varphi, \sin\theta\cos\varphi, \cos\theta) \end{aligned} \tag{29}$$

$$\boldsymbol{\omega} = (\dot{\psi}\sin\theta\sin\varphi + \dot{\theta}\cos\varphi, \dot{\psi}\sin\theta\cos\varphi - \dot{\theta}\sin\varphi, \dot{\psi}\cos\theta + \dot{\varphi}). \tag{30}$$

The problem considered here is the general problem of motion of a rigid body about a fixed point under the action of a combination of conservative potential and gyroscopic forces, described by the Lagrangian [17, 18],

$$L = \tfrac{1}{2}\boldsymbol{\omega}\mathbf{I}\cdot\boldsymbol{\omega} + \mathbf{l}\cdot\boldsymbol{\omega} - V \tag{31}$$

where $\mathbf{I} = \mathrm{diag}(A, B, C)$ is the inertia matrix of the body. The potential V and the vector $\mathbf{l}$ depend only on the Eulerian angles through the nine direction cosines α_1, α_2, α_3, β_1, β_2, β_3, γ_1, γ_2, γ_3.

The Lagrangian (31) describes a conservative system of three degrees of freedom, which admits the Jacobi integral (the Hamiltonian of the system)

$$I_1 \equiv H = \tfrac{1}{2}\boldsymbol{\omega}\mathbf{I}\cdot\boldsymbol{\omega} + V = \text{const.}$$

We note that for its complete integration in the sense of Liouville we must obtain two additional integrals of motion in involution with the Jacobi integral.

The equations of motion of a rigid body are usually written in the Euler–Poisson variables $\boldsymbol{\omega}, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}$. For the present problem this form, corresponding to (31), was derived in our work [18] to be

$$\begin{aligned}
&\dot{\boldsymbol{\omega}}\mathbf{I} + \boldsymbol{\omega}\times(\boldsymbol{\omega}\mathbf{I}+\boldsymbol{\mu}) = \boldsymbol{\alpha}\times\frac{\partial V}{\partial\boldsymbol{\alpha}} + \boldsymbol{\beta}\times\frac{\partial V}{\partial\boldsymbol{\beta}} + \boldsymbol{\gamma}\times\frac{\partial V}{\partial\boldsymbol{\gamma}}\\
&\dot{\boldsymbol{\alpha}} + \boldsymbol{\omega}\times\boldsymbol{\alpha} = \mathbf{0} \qquad \dot{\boldsymbol{\beta}} + \boldsymbol{\omega}\times\boldsymbol{\beta} = \mathbf{0} \qquad \dot{\boldsymbol{\gamma}} + \boldsymbol{\omega}\times\boldsymbol{\gamma} = \mathbf{0}
\end{aligned}\tag{32}$$

where $\mathbf{I}$ is the inertia tensor of the body at the fixed point and

$$\begin{aligned}
\boldsymbol{\mu} = \mathbf{l} + &\left(\boldsymbol{\alpha}\times\frac{\partial}{\partial\boldsymbol{\alpha}} + \boldsymbol{\beta}\times\frac{\partial}{\partial\boldsymbol{\beta}} + \boldsymbol{\gamma}\times\frac{\partial}{\partial\boldsymbol{\gamma}}\right)\times\mathbf{l}\\
&\equiv \frac{\partial}{\partial\boldsymbol{\alpha}}(\mathbf{l}\cdot\boldsymbol{\alpha}) + \frac{\partial}{\partial\boldsymbol{\beta}}(\mathbf{l}\cdot\boldsymbol{\beta}) + \frac{\partial}{\partial\boldsymbol{\gamma}}(\mathbf{l}\cdot\boldsymbol{\gamma}) - \left(\frac{\partial}{\partial\boldsymbol{\alpha}}\cdot\mathbf{l}\right)\boldsymbol{\alpha}\\
&\quad - \left(\frac{\partial}{\partial\boldsymbol{\beta}}\cdot\mathbf{l}\right)\boldsymbol{\beta} - \left(\frac{\partial}{\partial\boldsymbol{\gamma}}\cdot\mathbf{l}\right)\boldsymbol{\gamma} - 2\mathbf{l}.
\end{aligned}\tag{33}$$

It was shown in [17] that different terms of equations (32) in their general form may be interpreted in most cases in one or more of the following (or other) ways.

The potential V can be understood as being due to the scalar interactions of a gravitational field with the mass distribution in the body, an electric field with a permanent distribution of electric charges and a magnetic field with some magnetized parts or steady currents in the electric circuits on the body. A constant term $\boldsymbol{\sigma}$ of the vectors $\boldsymbol{\mu}$ and $\mathbf{l}$ is the so-called gyrostatic moment that appears when the body carries a symmetric rotor forced to rotate uniformly with respect to the principal body [19], while the variable terms of $\boldsymbol{\mu}$ may appear as a result of the Lorentz effect of the magnetic field on the electric charges. Let $\mathcal{B}$ and $\mathcal{A}$ be the intensity of the magnetic field and the vector potential of this field at the point $\mathbf{r}$ of the body where the current charge element $\mathrm{d}e$ is placed. In that case one can write the vector $\mathbf{l}$ as (for details see [17])[1]

$$\mathbf{l} = \boldsymbol{\sigma} + \int \mathbf{r}\times\mathcal{A}\,\mathrm{d}e \tag{34}$$

while $\boldsymbol{\mu}$ can be derived from $\mathbf{l}$ according to (33) or constructed directly in the form [17]

$$\begin{aligned}
\boldsymbol{\mu} &= \boldsymbol{\sigma} - \int(\mathbf{r}\cdot\mathcal{B})\mathbf{r}\,\mathrm{d}e\\
&= \boldsymbol{\sigma} + \int\left(\mathbf{r}\cdot\frac{\partial\Omega}{\partial\mathbf{r}}\right)\mathbf{r}\,\mathrm{d}e
\end{aligned}\tag{35}$$

where Ω is the scalar magnetic potential. In several cases of interest for future application, Ω can be expressed as a sum

$$\Omega(X, Y, Z) = \Omega_1(X, Y, Z) + \cdots + \Omega_N(X, Y, Z) \tag{36}$$

[1] Here MKS units are used. In Gaussian units $\mathrm{d}e$ should be divided by the velocity of light c (e.g. [20]). We also assume that velocity and acceleration are sufficiently small to neglect both relativistic effects and classical radiation damping.

of homogeneous harmonic polynomials up to the *N*th degree, the formula (33) can be replaced by

$$\boldsymbol{\mu} = \boldsymbol{\sigma} + \sum_{s=1}^{N} s \int \Omega_s(X, Y, Z)\mathbf{r}\, \mathrm{d}e.$$

Now, expressing $\mathbf{r}$ in the moving body axes xyz, we get

$$\boldsymbol{\mu} = \boldsymbol{\sigma} + \sum_{s=1}^{N} s \int \Omega_s(\mathbf{r}\cdot\boldsymbol{\alpha}, \mathbf{r}\cdot\boldsymbol{\beta}, \mathbf{r}\cdot\boldsymbol{\gamma})\mathbf{r}\, \mathrm{d}e. \tag{37}$$

i.e. components are polynomial in the direction cosines.

The gyrostatic moment $\boldsymbol{\sigma}$ can also be due to internal cyclic degrees of freedom such as the circulation of fluid in holes inside the body or to forced stationary motions such as motors and the flow of fluids in circuits in the body (see e.g.[21]). In an interesting alternative, due to Levi-Civita [22], the rotor is left to move freely around its axis of symmetry fixed in the body. In that case the matrix $\mathbf{I}$ is not simply the matrix of inertia of the system, but depends on the direction of the rotor in the body and on the cyclic constant of its motion.

It was noted in [17] that a variable part of the vector $\boldsymbol{\mu}$ also appears in the case of a moving dielectric body in a combination of electric and magnetic fields.

6. The case of a rigid body under axisymmetric forces

If all forces acting on the body have the *Z*-axis (say) as a common axis of symmetry, then V and $\mathbf{l}$ (and consequently $\boldsymbol{\mu}$) are functions of $\boldsymbol{\gamma}$ only and the angle ψ is a cyclic variable in the Lagrangian (31). Equations of motion take the familiar form used in [17]

$$\begin{aligned} &\dot{\boldsymbol{\omega}}\mathbf{I} + \boldsymbol{\omega} \times (\boldsymbol{\omega}\mathbf{I} + \boldsymbol{\mu}) = \boldsymbol{\gamma} \times \frac{\partial V}{\partial \boldsymbol{\gamma}} \\ &\dot{\boldsymbol{\gamma}} + \boldsymbol{\omega} \times \boldsymbol{\gamma} = \mathbf{0} \end{aligned} \tag{38}$$

$$\begin{aligned} \boldsymbol{\mu} &= \mathbf{l} + \left(\boldsymbol{\gamma} \times \frac{\partial}{\partial \boldsymbol{\gamma}}\right) \times \mathbf{l} \\ &= \frac{\partial}{\partial \boldsymbol{\gamma}}(\mathbf{l}\cdot\boldsymbol{\gamma}) - \left(\frac{\partial}{\partial \boldsymbol{\gamma}}\cdot\mathbf{l}\right)\boldsymbol{\gamma}. \end{aligned} \tag{39}$$

Only one Poisson equation is needed for $\boldsymbol{\gamma}$ in the system (38) to be closed. Having solved this system in any concrete case, the determination of the angle ψ and hence the vectors $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ come out in a natural way by means of a quadrature using the cyclic integral

$$(\boldsymbol{\omega}\mathbf{I} + \mathbf{l})\cdot\boldsymbol{\gamma} = f. \tag{40}$$

From (29) and (30) one can easily see that the cyclic velocity transformation in the equations of motion (38) is equivalent to a change in the angular velocity

$$\boldsymbol{\omega} = \boldsymbol{\omega}' + \nu(\boldsymbol{\gamma})\boldsymbol{\gamma}. \tag{41}$$

Applying this transformation in the sense of section 2 to (31) we get the Lagrangian

$$L' = \tfrac{1}{2}\boldsymbol{\omega}'\mathbf{I}\cdot\boldsymbol{\omega}' + \mathbf{l}'\cdot\boldsymbol{\omega}' - V'. \tag{42}$$

The equations of motion derived from the new Lagrangian are

$$\begin{aligned} &\dot{\boldsymbol{\omega}}'\mathbf{I} + \boldsymbol{\omega}' \times (\boldsymbol{\omega}'\mathbf{I} + \boldsymbol{\mu}') = \boldsymbol{\gamma} \times \frac{\partial V'}{\partial \boldsymbol{\gamma}} \\ &\dot{\boldsymbol{\gamma}} + \boldsymbol{\omega}' \times \boldsymbol{\gamma} = \mathbf{0} \end{aligned} \tag{43}$$

on the integral level

$$(\boldsymbol{\omega}'\mathbf{I} + \mathbf{l}') \cdot \boldsymbol{\gamma} = f \tag{44}$$

where

$$\begin{aligned} V' &= V + (f - \mathbf{l}\cdot\boldsymbol{\gamma})\nu - \tfrac{1}{2}\nu^2\boldsymbol{\gamma}\mathbf{I}\cdot\boldsymbol{\gamma} \\ \mathbf{l}' &= \mathbf{l} + \nu\boldsymbol{\gamma}\mathbf{I} \\ \boldsymbol{\mu}' &= \boldsymbol{\mu} - 2\nu\boldsymbol{\gamma}\bar{\mathbf{I}} \qquad \bar{\mathbf{I}} = \tfrac{1}{2}\,\mathrm{tr}\,(\mathbf{I})\boldsymbol{\delta} - \mathbf{I}. \end{aligned} \tag{45}$$

The system described by (43) and (44) is mathematically equivalent to that described by (38)–(40). The solution of one of them can readily be obtained from that of the other through relation (41). The two systems have the same solution with respect to the vector $\boldsymbol{\gamma}$ or to the Eulerian angles θ and φ. The angles of precession for the two systems are different by the amount

$$\psi - \psi' = \int_{t_0}^{t} \nu(\boldsymbol{\gamma}(t))\,\mathrm{d}t.$$

Each of the systems (43) and (38) can be interpreted on its own as representing the motion of a body under the forces given in each case with respect to the inertial frame.

The equivalence of the two systems for constant ν was noted earlier in [17], using directly the fact that transformation (41) preserves the form of the Euler–Poisson equations (38), changing only V to V' and $\mathbf{l}$ to $\mathbf{l}'$ or, equivalently, $\boldsymbol{\mu}$ to $\boldsymbol{\mu}'$. In that case the constant term $f\nu$ in the transformed potential is insignificant and can be omitted.

The invariance of equations of motion under transformation (41) for constant ν was used in [17, 23–25] to generate new generalizations of known integrable problems and to relate results concerning previously unrelated problems. In [9, 14] we have introduced six general and 15 conditional integrable problems of rigid body dynamics generalizing all the known integrable cases in the subject either by including a set of arbitrary constants or by an arbitrary scalar function of the vector $\boldsymbol{\gamma}$. The invariance of the Euler–Poisson equations under the transformation (41) for variable ν was used to that end. The method of the present paper can easily be seen to lead to the same results as in [9, 14] if applied to the dynamics of a rigid body under forces with a common space axis of symmetry. Those results will not be reproduced here.

7. Integrable cases of an axisymmetric body under asymmetric forces

In the present section we examine a type of symmetry that is compatible with the non-symmetric character of the fields applied to the body and allows for a cyclic integral to exist. For this type the body must admit axial dynamical symmetry, say $B = A$, such that the proper rotation angle φ is cyclic. As we will note below, unlike the case of axially symmetric fields combination, the Euler equations of motion in the present cases are *not* invariant under the transformation of the cyclic variable.

The potential can depend only on the angles θ, ψ or equivalently on the three direction cosines α_3, β_3, γ_3. The vector $\mathbf{l}$ should have the form

$$\mathbf{l} = (\lambda_1\gamma_1 + \lambda_2\gamma_2, \lambda_1\gamma_2 - \lambda_2\gamma_1, l_3)$$

where λ_1, λ_2, l_3 are dependent only on α_3, β_3, γ_3.

The Lagrangian can be written as

$$L = \tfrac{1}{2}[A(p^2 + q^2) + Cr^2] + \mathbf{l}\cdot\boldsymbol{\omega} - V(\theta, \psi) \tag{46}$$

and the cyclic integral has the form

$$Cr + l_3 = f.$$

The transformation of the cyclic variable $\dot{\varphi} = \dot{\varphi}' + \nu$, where $\nu = \nu(\alpha_3, \beta_3, \gamma_3)$, is equivalent to the change

$$r = r' + \nu. \tag{47}$$

In the transformed Lagrangian, V and $\mathbf{l}$ are transformed to

$$\begin{aligned} V' &= V + \nu(f - l_3) - \tfrac{1}{2}C\nu^2 \\ \mathbf{l}' &= \mathbf{l} + (0, 0, \nu C). \end{aligned}$$

The addition of the new terms to the Lagrangian does not affect in any way the integrability of the Lagrangian system nor would it complicate its explicit solution.

In particular, let us consider a system that moves under only potential forces, i.e. $\mathbf{l} = \mathbf{0}$. Taking $\nu = n =$ const, a constant gyroscopic moment Cn can always be added along the axis of symmetry of the body, so that the system remains integrable. The modified system will have the same general solution in terms of time as the original one with regard to the angles ψ, θ but the angle φ will differ by a constant rate n. In other words, this means that the dynamical effect of the rotor with gyrostatic moment Cn is equivalent to increasing the axial component of the principal body by the amount n.

We also note here that the present transformation of the problem of motion is not directly associated with an invariance property of the Euler–Poisson equations of motion as was the case for axially symmetric fields discussed in the preceding section and in full detail in [9].

There are only two known integrable cases of the type under consideration to which this result can be applied.

7.1. A new general integrable case

In a well-known general case due to Brun, the body moves under forces whose potential has a certain quadratic form in the nine direction cosines. The integrals of motion for that case were found by Brun [26] and integration of the equations of motion was carried out by Bogoyavlensky [27]. The restriction of this problem to the case of an axisymmetric body $A = B$ gives

$$\begin{aligned} V &= \tfrac{1}{2}\left(a\alpha_3^2 + b\beta_3^2 + c\gamma_3^2\right) \\ \mathbf{l} &= (0, 0, 0). \end{aligned} \tag{48}$$

If we transform this case using

$$\nu = n + n_1\alpha_3^2 + n_2\beta_3^2 + n_3\gamma_3^2 \tag{49}$$

we get a new integrable case in which

$$\begin{aligned} V &= \tfrac{1}{2}\left(a\alpha_3^2 + b\beta_3^2 + c\gamma_3^2\right) - \tfrac{1}{2}C\left(n + n_1\alpha_3^2 + n_2\beta_3^2 + n_3\gamma_3^2\right)^2 \\ \mathbf{l} &= \left(0, 0, C\left(n + n_1\alpha_3^2 + n_2\beta_3^2 + n_3\gamma_3^2\right)\right). \end{aligned} \tag{50}$$

This gives

$$\begin{aligned} \mu_1 &= -2C(n_1\alpha_1\alpha_3 + n_2\beta_1\beta_3 + n_3\gamma_1\gamma_3) \\ \mu_2 &= -2C(n_1\alpha_2\alpha_3 + n_2\beta_2\beta_3 + n_3\gamma_2\gamma_3) \\ \mu_3 &= C\left(n + n_1\alpha_3^2 + n_2\beta_3^2 + n_3\gamma_3^2\right). \end{aligned} \tag{51}$$

The integrals I_2, I_3 in this case can be written in the following form, which can be verified directly:

$$I_2 = C\left(r + n + n_1\alpha_3^2 + n_2\beta_3^2 + n_3\gamma_3^2\right)$$
$$\begin{aligned}I_3 = {} & (a - 2n_1 I_2)[A(p\alpha_1 + q\alpha_2) + C(r+\nu)\alpha_3]^2 + (b - 2n_2 I_2)[A(p\beta_1 + q\beta_2) + C(r+\nu)\beta_3]^2 \\ & + (c - 2n_3 I_2)[A(p\gamma_1 + q\gamma_2) + C(r+\nu)\gamma_3]^2 - A[(b - 2n_2 I_2)(c - 2n_3 I_2)\alpha_3^2 \\ & + (c - 2n_3 I_2)(a - 2n_1 I_2)\beta_3^2 + (a - 2n_1 I_2)(b - 2n_2 I_2)\gamma_3^2].\end{aligned} \tag{52}$$

Note that I_3 is of third degree in general. However, when $n_1 : n_2 : n_3 :: a : b : c$ a constant factor can be cancelled out and I_3 becomes of second degree. If we set $n_1 = n_2 = n_3 = 0$ then the new case generalizes the original one merely by the addition of a gyrostatic moment Cn along the axis of dynamical symmetry. The present case (50)–(52) also generalizes one, namely the second, of the two cases reported in [28] and reduces to it when $A = C$. It is to be noted that the cases of [28] were obtained by a completely different method.

7.2. *The second integrable case*

In [28], in a completely different context, an integrable case valid for a body of spherical dynamical symmetry $A = B = C$ was introduced, for which

$$\begin{aligned}V = {} & s_1\alpha_3 + s_2\beta_3 + s_3\gamma_3 - \frac{1}{2A}\left(bc\alpha_3^2 + ca\beta_3^2 + ab\gamma_3^2\right) - \frac{1}{2}A(n + n_1\alpha_3 + n_2\beta_3 + n_3\gamma_3)^2 \\ & + \frac{1}{2}(n + n_1\alpha_3 + n_2\beta_3 + n_3\gamma_3)\left[(b+c)\alpha_3^2 + (c+a)\beta_3^2 + (a+b)\gamma_3^2\right] \\ \mu_1 = {} & -[A(n_1\alpha_1 + n_2\beta_1 + n_3\gamma_1) + a\alpha_3\alpha_1 + b\beta_3\beta_1 + c\gamma_3\gamma_1] \\ \mu_2 = {} & -[A(n_1\alpha_2 + n_2\beta_2 + n_3\gamma_2) + a\alpha_3\alpha_2 + b\beta_3\beta_2 + c\gamma_3\gamma_2] \\ \mu_3 = {} & A(n + n_1\alpha_3 + n_2\beta_3 + n_3\gamma_3) - \left(a\alpha_3^2 + b\beta_3^2 + c\gamma_3^2\right).\end{aligned} \tag{53}$$

Alternatively, this case could be obtained by using $\nu = n + n_1\alpha_3 + n_2\beta_3 + n_3\gamma_3$ in transforming its special case $n = n_1 = n_2 = n_3 = 0$. The last case is the analogue of Lyapounov's case in the dynamics of a rigid body in a liquid [29] in the sense of [28].

8. Integrable cases of a body with combined symmetry

In our note [8] (see also [30]), one of the new integrable cases introduced concerned a heavy magnetized body-gyrostat with the Kovalevskaya configuration $A - B - 2C$ moving in a combination of uniform gravity and magnetic fields. In that case a linear integral of motion was found in the form of a sum of the projections of the angular momentum of the body in the Z- and z-directions. This integral was related to a cyclic variable $\psi + \varphi$ [30]. It was also shown that this integral results from the symmetry of the problem under rotation in one coordinate plane of the four-dimensional space of Euler's (Hamilton–Rodrigues') parameters [31] if used as configurational variables. Here we follow up the general case when the problem of motion admits this type of symmetry.

Let a rigid body-gyrostat with $A = B$ be in motion under the action of forces with potential $V(\theta, \psi - \varphi)$ and gyroscopic moments compatible with the present type of symmetry (this includes a gyrostatic moment σ directed along the axis of dynamical symmetry). The Lagrangian of this system is

$$L = \tfrac{1}{2}[A(p^2+q^2)+Cr^2]+\mathbf{l}\cdot\boldsymbol{\omega} - V$$
$$= \tfrac{1}{2}[A(\dot\theta^2+\sin^2\theta\dot\psi^2)+C(\dot\psi\cos\theta+\dot\varphi)^2]+\dot\psi\mathbf{l}\cdot\gamma+\dot\varphi l_3+\dot\theta(l_1\gamma_2-l_2\gamma_1)-V. \tag{54}$$

It has the cyclic variable $\psi+\varphi$ under the conditions that the functions $V, \mathbf{l}\cdot\gamma,\ l_3, l_1\gamma_2-l_2\gamma_1$ depend only on the variables θ and $\psi-\varphi$. In terms of the rotation matrix (see equation (29)) this allows only the combinations of the direction cosines

$$\alpha_1-\beta_2, \alpha_2+\beta_1, \gamma_3.$$

The corresponding integral is

$$A(p\gamma_1+q\gamma_2)+Cr\gamma_3+\mathbf{l}\cdot\gamma+Cr+l_3 = \text{const}.$$

The transformation $(\dot\psi,\dot\varphi)\to(\dot\psi+\nu,\dot\varphi+\nu)$ changes (p,q,r) to $(p+\nu\gamma_1, q+\nu\gamma_2, r+\nu(\gamma_3+1))$ and leads to the new pair

$$\begin{aligned} \mathbf{l}' &= \mathbf{l}+\nu(A\gamma_1, A\gamma_2, C(\gamma_3+1)) \\ V' &= V(\theta,\psi-\varphi)+\nu(f-\mathbf{l}\cdot\gamma-l_3)-\frac{\nu^2}{2}\left[A\left(\gamma_2^2+\gamma_1^2\right)+C(\gamma_3+1)^2\right]. \end{aligned} \tag{55}$$

The two systems described by the $(\mathbf{l}, V)$ and $(\mathbf{l}', V')$ are mathematically equivalent. From the physical point of view the latter system involves several changes to the first. The most interesting consequence of this equivalence is that any integrable case of (54) always generates a more general integrable case of (55) containing the additional function ν. Applying this to the three known cases of general integrability of (54) we obtain the following three new general cases.

8.1. Case 1: a body with the Kovalevskaya configuration

This case is obtained for $A = B = 2C$ by choosing

$$\nu = n+n_1(\alpha_1-\beta_2)+n_2(\alpha_2+\beta_1) \tag{56}$$

in (55) applied to the case of [8]. We get

$$\begin{aligned} &\mathbf{l} = C(2\nu\gamma_1, 2\nu\gamma_2, \sigma+\nu(1+\gamma_3)) \\ &\mu_1/C = -n\gamma_1+n_1(\alpha_2\gamma_2-2\alpha_1\gamma_1+3\beta_1\gamma_2)+n_2(\beta_2\gamma_2-2\beta_1\gamma_1-3\alpha_1\gamma_2) \\ &\mu_2/C = -n\gamma_2+n_1(-\beta_1\gamma_1+2\beta_2\gamma_2-3\alpha_2\gamma_1)+n_2(\alpha_1\gamma_1-2\alpha_2\gamma_2-3\beta_2\gamma_1) \\ &\mu_3/C = \sigma+n(1-3\gamma_3)+n_1[\beta_3\gamma_2-\alpha_3\gamma_1+4\gamma_3(\beta_2-\alpha_1)]-n_2[\alpha_3\gamma_2+\beta_3\gamma_1+4\gamma_3(\alpha_2+\beta_1)] \\ &V = C\left\{a_1(\alpha_1-\beta_2)+a_2(\alpha_2+\beta_1)-\frac{\nu^2}{2}\left[2\left(\gamma_2^2+\gamma_1^2\right)+(\gamma_3+1)^2\right]-\sigma\nu(1+\gamma_3)\right\}. \end{aligned} \tag{57}$$

The corresponding integrals are

$$\begin{aligned} I_2 &= 2(p\gamma_1+q\gamma_2)+r(1+\gamma_3)+\sigma\gamma_3+\nu(1+\gamma_3)(3-\gamma_3) \\ I_3 &= [(p+\nu\gamma_1)^2-(q+\nu\gamma_2)^2-(a_1-n_1I_2)(\alpha_1+\beta_2)+(a_2-n_2I_2)(\alpha_2-\beta_1)]^2 \\ &\quad +[2(p+\nu\gamma_1)(q+\nu\gamma_2)-(a_2-n_2I_2)(\alpha_1+\beta_2)-(a_1-n_1I_2)(\alpha_2-\beta_1)]^2 \\ &\quad +2\sigma[r-\sigma+\nu(1+\gamma_3)][(p+\nu\gamma_1)^2+(q+\nu\gamma_2)^2] \\ &\quad -4\sigma\{(p+\nu\gamma_1)[(a_1-n_1I_2)\alpha_3+(a_2-n_2I_2)\beta_3] \\ &\quad +(q+\nu\gamma_2)[(a_2-n_2I_2)\alpha_3-(a_1-n_1I_2)\beta_3]\}. \end{aligned} \tag{58}$$

The linear integral I_2 corresponds to the cyclic variable $\psi+\varphi$. The integral I_3, though it became more complicated, still has the fourth degree.

This case generalizes by the presence of the two parameters n_1, n_2 the case found in [32]. The explicit solution of the system (57), (58) can be deduced as described in the above sections from that of the special version $n = n_1 = n_2 = 0$. That is the case noted in [8, 30]. It is now known that the last case is also a special version of a case found in [33] in which no cyclic variable is present and which is solvable by the Lax pair method. The solution $\psi = \Psi(t)$, $\theta = \Theta(t)$, $\varphi = \Phi(t)$ of our case when $n = n_1 = n_2 = 0$ follows from that solution. The solution of the full system (57), (58) can readily be written as $\psi(t) = \Psi(t) - \int \nu(t)\,\mathrm{d}t, \theta(t) = \Theta(t), \varphi(t) = \Phi(t) - \int \nu(t)\,\mathrm{d}t$, where $\nu(t) = n + (1 - \cos\Theta(t))(n_1 \cos(\Psi(t) - \Phi(t)) + n_2 \sin(\Psi(t) - \Phi(t)))$. Note that the palpable coordinates θ and $\psi - \varphi$ are not affected by the extra parameters n, n_1 and n_2.

8.2. Case 2: a case of a dynamically spherical body

For the second integrable case of (54)

$$
\begin{aligned}
&A = B = C\\
&\mathbf{l} \;= \mathbf{0}\\
&V = v(\alpha_1 + \beta_2 + \gamma_3)\\
&\quad = v((1 + \cos\theta)\cos(\psi + \varphi) + \cos\theta)
\end{aligned}
\tag{59}
$$

where v is an arbitrary function of its argument. The integrability, separation of variables and explicit solution of this problem were presented in [34]. According to the above considerations this case admits the following integrable generalization corresponding to the cyclic variable $\psi - \varphi$:

$$
\begin{aligned}
V \;&= v(\alpha_1 + \beta_2 + \gamma_3) - A\nu^2(1 - \gamma_3)\\
\mu_1 &= -A\gamma_1\nu + A[(\gamma_3 - 1)(\alpha_3 - \gamma_1) - \gamma_2(\beta_1 - \alpha_2)]\nu'\\
\mu_2 &= -A\gamma_2\nu + A[(\gamma_3 - 1)(\beta_3 - \gamma_2) + \gamma_1(\beta_1 - \alpha_2)]\nu'\\
\mu_3 &= -A(1 + \gamma_3)\nu + A(1 - \gamma_3)(1 + \alpha_1 + \beta_2 + \gamma_3)\nu'
\end{aligned}
\tag{60}
$$

where ν' is the derivative of ν with respect to its argument. Integrals of motion can be shown to be

$$
\begin{aligned}
I_2 &= \boldsymbol{\omega}\cdot\boldsymbol{\gamma} - r + 2\nu(1 - \gamma_3)\\
I_3 &= [\boldsymbol{\omega}\cdot\boldsymbol{\alpha} - p - \nu(\gamma_1 + \alpha_3)]^2 + [\boldsymbol{\omega}\cdot\boldsymbol{\beta} - q - \nu(\gamma_2 + \beta_3)]^2\\
&\quad + [\boldsymbol{\omega}\cdot\boldsymbol{\gamma} - r + 2\nu(1 - \gamma_3)]^2.
\end{aligned}
\tag{61}
$$

The separation of variables can be achieved through a transformation to the Euler (Hamilton–Rodrigues) variables in the same way as in [34]. However, the solution can be obtained directly from that of the original problem according to the change $\theta(t), \psi(t), \varphi(t) \to \theta(t), \psi(t) + \int \nu(t)\mathrm{d}t, \varphi(t) - \int \nu(t)\mathrm{d}t$, where $\nu(t)$ stands for the value of ν in the original solution, i.e. $\nu(t) = \nu(\theta(t), \psi(t) + \varphi(t))$.

8.3. Case 3: another case of a dynamically spherical body

In [35] Bogoyavlensky noted that the dynamics of the body of the spherical dynamical symmetry $A = B = C$ is integrable for a potential which is linear in all the nine direction cosines. It can be written as

$$
V = \mathbf{a}\cdot\boldsymbol{\alpha} + \mathbf{b}\cdot\boldsymbol{\beta} + \mathbf{c}\cdot\boldsymbol{\gamma}
\tag{62}
$$

where **a**, **b** and **c** are arbitrary vectors constant in the body. This problem was reduced to the Neumann problem on the sphere S^3 [35]. The integrals of motion in that case are all quadratic in the angular velocities and the explicit solution can be found in terms of the Riemannian theta-functions as follows from that of the Neumann system [36, 37].

It is possible, without loss of generality, to choose the space and body axes to reduce (62) to the much simpler form

$$V = A(a\alpha_1 + b\beta_2 + c\gamma_3) \tag{63}$$

containing only three parameters. In fact, it can easily be seen that the potential (62) is a quadratic form in the four-dimensional space of the Euler (Hamilton–Rodrigues) parameters (e.g. [31, 38]). Through a rotation of the basis of the four-dimensional space, which is equivalent to the two rotations of the three-dimensional space and body systems, diagonalization of the quadratic form is possible and leads to the form (63). This simple form of the potential enables us to express the integrals of the problem in the Euler–Poisson variables, which was not done in [35]. We get

$$\begin{aligned}
i_2 &= \tfrac{1}{2}[ap(p\alpha_1 + q\alpha_2 + r\alpha_3) + bq(p\beta_1 + q\beta_2 + r\beta_3) + cr(p\gamma_1 + q\gamma_2 + r\gamma_3)] \\
&\quad + bc\alpha_1 + ca\beta_2 + ab\gamma_3 \\
i_3 &= \tfrac{1}{2}\left\{a^2\left[p^2 + (p\alpha_1 + q\alpha_2 + r\alpha_3)^2\right] + b^2\left[q^2 + (p\beta_1 + q\beta_2 + r\beta_3)^2\right]\right. \\
&\quad \left. + c^2\left[r^2 + (p\gamma_1 + q\gamma_2 + r\gamma_3)^2\right]\right\} + bcp(p\alpha_1 + q\alpha_2 + r\alpha_3) \\
&\quad + caq(p\beta_1 + q\beta_2 + r\beta_3) + abr(p\gamma_1 + q\gamma_2 + r\gamma_3) \\
&\quad + 2a(b^2 + c^2)\alpha_1 + 2b(c^2 + a^2)\beta_2 + 2c(a^2 + b^2)\gamma_3.
\end{aligned} \tag{64}$$

For our purpose, we need to find the condition on the parameters a, b and c that makes the system admit a linear integral. One can easily verify that this happens when two of those parameters are equal in modulus. Without loss of generality we can choose $b = -a$ and hence the potential becomes

$$V = A\{a(\alpha_1 - \beta_2) + c\gamma_3\}. \tag{65}$$

This choice makes the variable $\psi - \varphi$ cyclic and leads to the linear integral

$$I_2 = \boldsymbol{\omega} \cdot \boldsymbol{\gamma} + r. \tag{66}$$

Thus, applying the present method of generalization with $\nu = n + n_1(\alpha_1 - \beta_2) + n_2\gamma_3$, we arrive at a new case characterized by

$$V = A\{a(\alpha_1 - \beta_2) + c\gamma_3 - (1 + \gamma_3)[n + n_1(\alpha_1 - \beta_2) + n_2\gamma_3]^2\} \tag{67}$$

$$\begin{aligned}
\mu_1 &= A[-n\gamma_1 + 2n_1(\beta_1\gamma_2 - \alpha_1\gamma_1) - n_2\gamma_1(1 + 2\gamma_3)] \\
\mu_2 &= A[-n\gamma_2 + 2n_1(\beta_2\gamma_2 - \alpha_2\gamma_1) - n_2\gamma_1(1 + 2\gamma_3)] \\
\mu_3 &= A[n(1 - \gamma_3) - 2n_1\gamma_3(\alpha_1 - \beta_2) + n_2(1 - \gamma_3)(1 + 2\gamma_3)]
\end{aligned} \tag{68}$$

$$I_2 = \boldsymbol{\omega} \cdot \boldsymbol{\gamma} + r + 2(1 + \gamma_3)[n + n_1(\alpha_1 - \beta_2) + n_2\gamma_3] \tag{69}$$

$$\begin{aligned}
I_3 &= (a - n_1I_2)[(p + \nu\gamma_1)(\boldsymbol{\omega} \cdot \boldsymbol{\alpha} + \nu\alpha_3) - (q + \nu\gamma_2)(\boldsymbol{\omega} \cdot \boldsymbol{\beta} + \nu\beta_3)] \\
&\quad + (c - n_2I_2)[r + \nu(\gamma_3 + 1)][\boldsymbol{\omega} \cdot \boldsymbol{\gamma} + \nu(1 + \gamma_3)] \\
&\quad - 2(a - n_1I_2)(c - n_2I_2)(\alpha_1 - \beta_2) - 2(a - n_1I_2)^2\gamma_3
\end{aligned} \tag{70}$$

which contains the extra parameters n, n_1 and n_2. Note that the integral I_3 is a polynomial of the third degree in the angular velocities. It reduces to a quadratic form when either $n_1 : n_2 :: a : c$ or $n_1 = n_2 = 0$. As the case $n = n_1 = n_2 = 0$ reduces to a special version of Bogoyavlensky's case, the general solution of the full case (67)–(70) can also be expressed in terms of the hyperelliptic theta-functions.

In conclusion, we note that an application similar to that of the last three sections can be performed for other problems of rigid body dynamics with one or more cyclic variables. Examples are the problem of motion of a gyroscope in the Cardan suspension (e.g. [19]) and the problem of motion of a body with an ellipsoidal cavity filled with an ideal incompressible fluid in a state of vortex motion [40].

9. Examples of physical interpretation

From the considerations of section 5, we see that a physical interpretation of the obtained cases is possible within the framework of motion of charged, magnetized bodies in the presence of a non-uniform combination of the three classical fields. Due to the abundance of physical parameters representing the three distributions and the coefficients of the three potentials, it should be easy to adjust these parameters to match the potential in each case and, moreover, in a variety of choices.

We will carry out detailed examples of the less obvious adjustment of the scalar magnetic potential Ω and the charge distribution to meet the Lorentz effect giving rise to the vector $\boldsymbol{\mu}$ in each case. This will be done for the two cases of section 7. It is easy to verify that the most general harmonic second-degree polynomial potential can be reduced by a rotation transformation to the form

$$\Omega = a_1 X + a_2 Y + a_3 Z + \tfrac{1}{2}\left(a_{11}X^2 + a_{22}Y^2 + a_{33}Z^2\right) \tag{71}$$

with coefficients subject to the single condition

$$a_{11} + a_{22} + a_{33} = 0 \tag{72}$$

ensuring that Ω is harmonic. According to (37) we can write

$$\begin{aligned}
\mu_1 &= \int x F(\mathbf{r}\cdot\boldsymbol{\alpha}, \mathbf{r}\cdot\boldsymbol{\beta}, \mathbf{r}\cdot\boldsymbol{\gamma})\,\mathrm{d}e \\
\mu_2 &= \int y F(\mathbf{r}\cdot\boldsymbol{\alpha}, \mathbf{r}\cdot\boldsymbol{\beta}, \mathbf{r}\cdot\boldsymbol{\gamma})\,\mathrm{d}e \\
\mu_3 &= \boldsymbol{\sigma} + \int z F(\mathbf{r}\cdot\boldsymbol{\alpha}, \mathbf{r}\cdot\boldsymbol{\beta}, \mathbf{r}\cdot\boldsymbol{\gamma})\,\mathrm{d}e
\end{aligned} \tag{73}$$

where $\boldsymbol{\sigma}$ is a gyrostatic moment directed along the z-axis and

$$F(X, Y, Z) = a_1 X + a_2 Y + a_3 Z + a_{11}X^2 + a_{22}Y^2 + a_{33}Z^2. \tag{74}$$

To guarantee that φ is cyclic, we assume the distribution of electric charges on the body to be axisymmetric around the z-axis. By virtue of symmetry $\int x^{\varepsilon_1} y^{\varepsilon_2} z^{\varepsilon_3}\,\mathrm{d}e$ is symmetric in ε_1 and ε_2 and it vanishes whenever ε_1 or ε_2 is odd. We denote the remaining integrals as

$$\begin{aligned}
&\int x^2\,\mathrm{d}e = \int y^2\,\mathrm{d}e = J \qquad && \int z^2\,\mathrm{d}e = J' \\
&\int x^2 z\,\mathrm{d}e = \int y^2 z\,\mathrm{d}e = K \qquad && \int z^3\,\mathrm{d}e = K'.
\end{aligned}$$

We finally find

$$\begin{aligned}
\mu_1 &= J(a_1\alpha_1 + a_2\beta_1 + a_3\gamma_1) + K(a_{11}\alpha_3\alpha_1 + a_{22}\beta_3\beta_1 + a_{33}\gamma_3\gamma_1) \\
\mu_2 &= J(a_1\alpha_2 + a_2\beta_2 + a_3\gamma_2) + K(a_{11}\alpha_3\alpha_2 + a_{22}\beta_3\beta_2 + a_{33}\gamma_3\gamma_2) \\
\mu_3 &= \sigma + J'(a_1\alpha_3 + a_2\beta_3 + a_3\gamma_3) + K'\left(a_{11}\alpha_3^2 + a_{22}\beta_3^2 + a_{33}\gamma_3^2\right).
\end{aligned} \tag{75}$$

9.1. For case 7.1

Comparing (51)–(75) we find that the uniform part $-(a_1, a_2, a_3)$ of the external magnetic field must vanish, so that the potential (71) should be homogeneous quadratic. In addition to the symmetry around the z-axis, the charge distribution should satisfy the single condition $K' = -\frac{1}{2}K$, i.e.

$$\int (x^2 + 2z^2)z\,\mathrm{d}e = 0. \tag{76}$$

The coefficients in (51) in terms of the parameters of the body and field have the form

$$n = \frac{\sigma}{C} \qquad n_1 = -\frac{Ka_{11}}{2C} \qquad n_2 = -\frac{Ka_{22}}{2C} \qquad n_3 = -\frac{Ka_{33}}{2C}. \tag{77}$$

9.2. For case 7.2

Equating the coefficients in (53) and (75), we find that the charge distribution must satisfy the following two conditions $J' = -J$, $K' = K$, i.e.

$$\int z^2\,\mathrm{d}e = -\int x^2\,\mathrm{d}e \tag{78}$$

$$\int x^2 z\,\mathrm{d}e = \int z^3\,\mathrm{d}e \tag{79}$$

and, without any loss of generality, we can express the parameters in (53) in terms of the parameters of the body and external magnetic field in the form

$$\begin{aligned} n_1 &= -\frac{Ja_1}{A} \qquad n_2 = -\frac{Ja_2}{A} \qquad n_3 = -\frac{Ja_3}{A} \\ n &= \frac{\sigma}{A} \\ a &= -Ka_{11} \qquad b = -Ka_{22} \qquad c = -Ka_{33}. \end{aligned} \tag{80}$$

If we define the moments of inertia of the distribution by A_e, B_e, C_e, from symmetry we have $A_e = B_e$. The condition (78) imposed on the second moments of the charge distribution can be put in the form $A_e = B_e = 0$. This is not a serious restriction, since electric charge distribution, unlike mass, can take positive and negative densities.

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